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Effect of couple-stresses on the elastic bending of beams

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Abstract

The problem of the pure bending of a circular cylinder is solved within the linear couple-stress theory. The solution is obtained by correcting the classical solution with a solution in plane strain within the section. A generalized formula is thus derived for the bending inertia of a circular cross-section.

1. Introduction

The linear couple-stress theory is the simplest possible type of generalisation of the classical continuum theory: by allowing for possible effects of the rotation-gradient in addition to the strain, the number of elastic constants for an isotropic material is increased from two to four. It is a special case of the Cosserat (or micropolar) theory in which the micro-rotation field is treated as independent variables so that six elastic coefficients are required. Further generalisation may be achieved by allowing for possible effects of more general and higher strain-gradient, with or without independent micro-displacement field: the first strain-gradient theory requires seven elastic constants (Mindlin and Eshel, 1968; Germain, 1973), eighteen if the micro-displacement field is treated as independent variable (Mindlin, 1964), as much as the second strain-gradient theory does (Mindlin, 1965).

Within the framework of the linear couple-stress theory, a series of well-known classical problems of elasticity can be solved in a more or less simple manner (Mindlin and Tiersten, 1962; Mindlin, 1963; Koiter, 1964; Sokolowski, 1970). The closed-form solutions derived for problems such as the torsion of circular cylinders, the cylindrical bending of plates or the stress concentrations around circular holes, exhibit fundamental differences with respect to the classical solutions. In particular, they may explain/predict the so-called size-effect whereby the smaller is the size of the specimen, the stronger is its response.

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The purpose of this paper is to solve the problem of the pure bending of a circular cylinder, thus allowing for an ulterior assessment of the practical significance of the theory. After recalling the fundamental equations of the linear couple-stress theory, the problem of the pure bending of a circular cylinder is formulated and solved in two steps. First, the classical solution is injected into the system of equations and is found to violate one boundary condition. Then, a corrective solution is sought in plane strain within the section. From the complete solution, a generalized formula is finally derived for the bending inertia of a circular cross-section.

2. Fundamental equations of the linear couple-stress theory

In the linear couple-stress theory, the work of the internal forces is assumed to depend on the rotation-gradient, in addition to the strain (Mindlin and Tiersten, 1962; Mindlin, 1963; Koiter, 1964; Sokolowski, 1970). Instead of merely recalling the fundamental equations, the author felt it appropriate to rederive them through the principle of virtual work following the presentation of Germain (1973), that is starting from the expression of the work of the internal forces. The fundamental equations, in particular the boundary conditions, are thus obtained without any ambiguity. Furthermore, neither the skew-symmetric part of the stress tensor nor the trace of the couple-stress tensor need to be introduced. At the end, the constitutive equations are recalled in the case of a linear elastic isotropic medium.

2.1. Kinematics

The kinematic variables to be taken into account within the linear couple-stress theory are the strain $\boldsymbol{\varepsilon}$ and the gradient of rotation $\boldsymbol{\kappa}$:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}) = \varepsilon_{ji}$$

$$\kappa_{ij} = \tfrac{1}{2}e_{jkl}u_{l,ki} \quad (\kappa_{ii} = 0) \tag{1}$$

where $\boldsymbol{e}$ is the alternator. Other relevant quantities are the rotation tensor $\boldsymbol{\omega}$ and the rotation vector $\boldsymbol{w}$:

$$\omega_{ij} = \tfrac{1}{2}(u_{j,i} - u_{i,j}) = -\omega_{ji}$$

$$w_i = \tfrac{1}{2}e_{ijk}u_{k,j} \tag{2}$$

Eliminating the displacement $\boldsymbol{u}$ in (1) and (2), the following relations are obtained:

$$\omega_{ij} = e_{ijk}w_k$$

$$w_i = \tfrac{1}{2}e_{ijk}\omega_{jk}$$

$$\kappa_{ij} = w_{j,i} = \tfrac{1}{2}e_{jkl}\omega_{kl,i} \tag{3}$$

It should be mentioned that the definition of the gradient of rotation is not unique in the literature: in (1), the definition of Mindlin and Eshel (1968) has been chosen but some other authors (Koiter, 1964; Sokolowski, 1970; Germain, 1973) use the conjugate of κ_{ij} instead.

2.2. Work of the internal forces

The virtual work of the internal forces $\mathscr{W}_{(i)}$ is assumed to depend on the strain $\boldsymbol{\varepsilon}$ and on the gradient of rotation $\boldsymbol{\kappa}$, so that the associated stress quantities are the classical Cauchy stress tensor $\boldsymbol{\sigma}$ (symmetric) and the so-called couple-stress tensor $\boldsymbol{\mu}$ (deviatoric):

$$\mathscr{W}_{(i)} = -\int_\Omega \left(\sigma_{ij}\varepsilon_{ij} + \mu_{ij}\kappa_{ij}\right) \mathrm{d}v \tag{4}$$

In the presentation of Koiter (1964) also followed by Sokolowski (1970), expression (4) is not assumed a priori but derived from the Cosserat equations of equilibrium expressed in terms of the full (non-symmetric) stress tensor and the full (non-deviatoric) couple stress tensor. As a matter of fact, it may be shown that neither the skew-symmetric part of the stress tensor nor the first invariant of the couple-stress tensor contribute to $\mathscr{W}_{(i)}$.

Substituting $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ from (1) and using the divergence theorem, $\mathscr{W}_{(i)}$ may be transformed into

$$\begin{aligned}
\mathscr{W}_{(i)} &= -\int_\Omega \left(\sigma_{ij}u_{j,i} + \frac{1}{2}e_{jkl}\mu_{ij}u_{l,ki}\right) \mathrm{d}v \\
&= \int_\Omega \left(\sigma_{ij,i}u_j - \left(\sigma_{ij}u_j\right)_{,i} + \frac{1}{2}e_{jkl}\mu_{ij,i}u_{l,k} - \frac{1}{2}e_{jkl}\left(\mu_{ij}u_{l,k}\right)_{,i}\right) \mathrm{d}v \\
&= \int_\Omega \left(\sigma_{ij,i}u_j - \left(\sigma_{ij}u_j\right)_{,i} - \frac{1}{2}e_{jkl}\mu_{ij,ik}u_l + \frac{1}{2}e_{jkl}\left(\mu_{ij,i}u_l\right)_{,k} - \frac{1}{2}e_{jkl}\left(\mu_{ij}u_{l,k}\right)_{,i}\right) \mathrm{d}v \\
&= \int_\Omega \left(\sigma_{ij,i}u_j - \frac{1}{2}e_{jkl}\mu_{ij,ik}u_l\right) \mathrm{d}v + \int_{\partial\Omega}\left(-\sigma_{ij}u_jn_i + \frac{1}{2}e_{jkl}\mu_{ij,i}u_ln_k\right) \mathrm{d}s - \int_{\partial\Omega}\frac{1}{2}e_{jkl}\mu_{ij}u_{l,k}n_i \,\mathrm{d}s
\end{aligned} \tag{5}$$

where $\boldsymbol{n}$ is the outward unit normal to the bounding surface $\partial\Omega$. Introducing the component of the rotation vector tangent to $\partial\Omega$, i.e.

$$\tilde{w}_j = w_j - w_pn_pn_j = \tfrac{1}{2}e_{jkl}u_{l,k} - \tfrac{1}{2}e_{pkl}u_{l,k}n_pn_j \tag{6}$$

and the component of $\mu_{ij}n_i$ normal to $\partial\Omega$

$$\mu_{\underline{nn}} = \mu_{ij}n_in_j \tag{7}$$

the last integral in the right-hand-side of (5) becomes

$$\begin{aligned}
-\int_{\partial\Omega}\frac{1}{2}e_{jkl}\mu_{ij}u_{l,k}n_i \,\mathrm{d}s &= -\int_{\partial\Omega}\left(\left(\mu_{ij}n_i - \mu_{\underline{nn}}n_j\right)\tilde{w}_j + \frac{1}{2}e_{pkl}u_{l,k}n_pn_j\mu_{ij}n_i\right)\mathrm{d}s \\
&= -\int_{\partial\Omega}\left(\left(\mu_{ij}n_i - \mu_{\underline{nn}}n_j\right)\tilde{w}_j + \frac{1}{2}e_{pkl}\mu_{\underline{nn}}u_{l,k}n_p\right)\mathrm{d}s \\
&= -\int_{\partial\Omega}\left(\left(\mu_{ij}n_i - \mu_{\underline{nn}}n_j\right)\tilde{w}_j - \frac{1}{2}e_{pkl}\mu_{\underline{nn},k}u_ln_p\right)\mathrm{d}s - \int_{\partial\Omega}\frac{1}{2}e_{pkl}\left(\mu_{\underline{nn}}u_l\right)_{,k}n_p \,\mathrm{d}s
\end{aligned} \tag{8}$$

Making use of the Stokes theorem on a surface S having a closed boundary ∂S,

$$\int_S e_{plk} V_{l,k} n_p \, \mathrm{d}s = \oint_{\partial S} V_j t_j \, \mathrm{d}l \tag{9}$$

the last term in the right-hand-side of (8) is transformed into

$$-\int_{\partial\Omega} \frac{1}{2} e_{plk} (\mu_{\underline{nn}} u_l)_{,k} n_p \, \mathrm{d}s = -\int_\Gamma \frac{1}{2} [[\mu_{\underline{nn}} u_j]] t_j \, \mathrm{d}l = -\int_\Gamma \frac{1}{2} [[\mu_{\underline{nn}}]] u_j t_j \, \mathrm{d}l \tag{10}$$

Here, Γ is an edge of $\partial\Omega$ at which intersect two surfaces of different outward unit normals $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$. Furthermore, $\boldsymbol{t}$ is the unit vector tangent to Γ, oriented positively with respect to $\boldsymbol{n}^+$, i.e. such that the scalar triple product $(\boldsymbol{n}^+ \wedge \boldsymbol{n}^-) \cdot \boldsymbol{t}$ is positive. The double brackets indicate the jump of the enclosed quantity across the edge, i.e.

$$[[\mu_{\underline{nn}}]] = \mu_{ij} n_i^+ n_j^+ - \mu_{ij} n_i^- n_j^- \tag{11}$$

Substituting (8) and (10) into (5) leads to

$$\mathscr{W}_{(i)} = \int_\Omega \left(\sigma_{ij,i} - \frac{1}{2} e_{ikj} \mu_{pi,pk} \right) u_j \, \mathrm{d}v - \int_{\partial\Omega} \left(\sigma_{ij} + \frac{1}{2} e_{ijk} (\mu_{pk,p} - \mu_{\underline{nn},k}) \right) n_i u_j \, \mathrm{d}s$$

$$- \int_{\partial\Omega} (\mu_{ij} n_i - \mu_{\underline{nn}} n_j) \tilde{w}_j \, \mathrm{d}s - \int_\Gamma \frac{1}{2} [[\mu_{\underline{nn}}]] t_j u_j \, \mathrm{d}l \tag{12}$$

2.3. *Work of the external loads*

The external loads are of two types: the body loads and the contact loads. The former are composed of body forces $\boldsymbol{f}$ and body couples $\boldsymbol{c}$ so that their virtual work is

$$\mathscr{W}_{(b)} = \int_\Omega \left(f_j u_j + c_j w_j \right) \mathrm{d}v \tag{13}$$

The latter are composed of surface forces $\boldsymbol{p}$ and surface couples $\boldsymbol{q}$ on $\partial\Omega$ and line forces $\boldsymbol{P}$ on Γ, so that their virtual work is

$$\mathscr{W}_{(c)} = \int_{\partial\Omega} (p_j u_j + q_j w_j) \, \mathrm{d}s + \int_\Gamma P_j u_j \, \mathrm{d}l \tag{14}$$

Here, the allowance for line forces is motivated by the existence of the line integral in (12). Substituting $\boldsymbol{w}$ from (1) and using the divergence theorem, (13) may be transformed into

$$\mathscr{W}_{(b)} = \int_\Omega \left(f_j u_j + \frac{1}{2} e_{jkl} c_j u_{l,k} \right) \mathrm{d}v$$

$$= \int_\Omega \left(f_j u_j - \frac{1}{2} e_{jkl} c_{j,k} u_l + \frac{1}{2} e_{jkl} (c_j u_l)_{,k} \right) \mathrm{d}v$$

$$= \int_\Omega \left(f_j - \frac{1}{2} e_{jpk} c_{p,k} \right) u_j \, \mathrm{d}v + \int_{\partial\Omega} \frac{1}{2} e_{jpk} c_p n_k u_j \, \mathrm{d}s \tag{15}$$

Decomposing $\boldsymbol{w}$ according to (6), introducing the normal component of $\boldsymbol{q}$

$$q_{\underline{n}} = q_i n_i \tag{16}$$

and using the Stokes theorem, (14) may be transformed into

$$\begin{aligned}
\mathscr{W}_{(c)} &= \int_{\partial\Omega} \left(p_j u_j + (q_j - q_{\underline{n}} n_j)\tilde{w}_j + \frac{1}{2} e_{pkl} n_p n_j q_j u_{l,k} \right) \mathrm{d}s + \int_\Gamma P_j u_j \, \mathrm{d}l \\
&= \int_{\partial\Omega} \left(p_j u_j + (q_j - q_{\underline{n}} n_j)\tilde{w}_j + \frac{1}{2} e_{pkl} n_p q_{\underline{n}} u_{l,k} \right) \mathrm{d}s + \int_\Gamma P_j u_j \, \mathrm{d}l \\
&= \int_{\partial\Omega} \left(p_j u_j + (q_j - q_{\underline{n}} n_j)\tilde{w}_j - \frac{1}{2} e_{pkl} q_{\underline{n},k} u_l n_p + \frac{1}{2} e_{pkl} (q_{\underline{n}} u_l)_{,k} n_p \right) \mathrm{d}s + \int_\Gamma P_j u_j \, \mathrm{d}l \\
&= \int_{\partial\Omega} \left(\left(p_j + \frac{1}{2} e_{jkp} q_{\underline{n},k} n_p \right) u_j + (q_j - q_{\underline{n}} n_j)\tilde{w}_j \right) \mathrm{d}s + \int_\Gamma \left(P_j + \frac{1}{2} [[q_{\underline{n}}]] t_j \right) u_j \, \mathrm{d}l
\end{aligned} \tag{17}$$

By introducing the following reduced contact loads

$$\begin{aligned}
\bar{p}_i &= p_i + \tfrac{1}{2} e_{ijk} q_{\underline{n},j} n_k \\
\bar{q}_i &= q_i - q_{\underline{n}} n_i \\
\bar{P}_i &= P_i + \tfrac{1}{2} [[q_{\underline{n}}]] t_i
\end{aligned} \tag{18}$$

$\mathscr{W}_{(c)}$ is simply written as

$$\mathscr{W}_{(c)} = \int_{\partial\Omega} \left(\bar{p}_j u_j + \bar{q}_j \tilde{w}_j \right) \mathrm{d}s + \int_\Gamma \bar{P}_j u_j \, \mathrm{d}l \tag{19}$$

At each point of $\partial\Omega$, the contact loads can thus be represented by only five independent quantities, the three componentss of $\bar{\boldsymbol{p}}$ and the two components of the tangential vector $\bar{\boldsymbol{q}}$. As a matter of fact, the normal component of the surface couple $q_{\underline{n}}$ does not contribute directly to $\mathscr{W}_{(c)}$ but indirectly through the reduced line forces and the tangential components of the reduced surface forces. This is due to the fact that the kinematic counterpart of $q_{\underline{n}}$, i.e. the normal component of the rotation, is fully specified by the distribution of tangential displacements over the boundary. As a consequence, the number of boundary conditions holding on a smooth surface is not six but only five. As mentioned by Koiter (1964), a very similar situation is encountered in the bending theory of plates where the number of boundary conditions are reduced from three to two.

Eqns (14) and (19) show that the reduced contact loads are equivalent to the original ones in an energetic sense. Consequently, the resultant $\mathscr{R}$ and the moment resultant $\mathscr{M}$ of the loads acting on a portion S of the boundary may be computed indifferently from the original or reduced loads. Indeed, $\mathscr{R}$ and $\mathscr{M}$ are, by definition, the dual quantities of the translation $\boldsymbol{U}$ and the rotation $\boldsymbol{W}$ in the work of the loads during a rigid displacement of S. Introducing this rigid displacement i.e.

$$u_j = U_j + e_{jkl} W_k x_l \tag{20}$$

in (14) or (19), and reducing the integral to S and to its edges Γ_S, one gets

$$\mathscr{W}=\left(\int_S p_j\,\mathrm{d}s+\int_{\Gamma_S}P_j\,\mathrm{d}l\right)U_i+\left(\int_S e_{ijk}x_jp_k\,\mathrm{d}s+\int_{\Gamma_s}e_{ijk}x_jP_k\,\mathrm{d}l+\int_S q_i\,\mathrm{d}s\right)W_i$$

$$=\left(\int_S \bar{p}_j\,\mathrm{d}s+\int_{\Gamma_S}\bar{P}_j\,\mathrm{d}l\right)U_i+\left(\int_S e_{ijk}x_j\bar{p}_k\,\mathrm{d}s+\int_{\Gamma_S}e_{ijk}x_j\bar{P}_k\,\mathrm{d}l+\int_S \bar{q}_i\,\mathrm{d}s\right)W_i \tag{21}$$

so that

$$\mathscr{R}_i=\int_S p_j\,\mathrm{d}s+\int_{\Gamma_S}P_j\,\mathrm{d}l=\int_S \bar{p}_j\,\mathrm{d}s+\int_{\Gamma_S}\bar{P}_j\,\mathrm{d}l$$

$$\mathscr{M}_i=\int_S e_{ijk}x_jp_k\,\mathrm{d}s+\int_{\Gamma_S}e_{ijk}x_jP_k\,\mathrm{d}l+\int_S q_i\,\mathrm{d}s=\int_S e_{ijk}x_j\bar{p}_k\,\mathrm{d}s+\int_{\Gamma_S}e_{ijk}x_j\bar{P}_k\,\mathrm{d}l+\int_S \bar{q}_i\,\mathrm{d}s \tag{22}$$

2.4. Equilibrium equations and boundary conditions in stresses

The principle of the virtual work ensures that $\mathscr{W}_{(i)}+\mathscr{W}_{(b)}+\mathscr{W}_{(c)}$ is zero for any virtual displacement $\boldsymbol{u}$. From (12), (15) and (19), we get

$$\int_\Omega\left(\sigma_{ij,i}-\frac{1}{2}e_{ijk}\left(\mu_{pk,pi}+c_{k,i}\right)+f_j\right)u_j\,\mathrm{d}v+\int_{\partial\Omega}\left(\left(-\sigma_{ij}+\frac{1}{2}e_{ijk}\left(\mu_{pk,p}+c_k-\mu_{\underline{nn},k}\right)\right)n_i+\bar{p}_j\right)u_j\,\mathrm{d}s$$

$$+\int_{\partial\Omega}\left(-\mu_{ij}n_i+\mu_{\underline{nn}}n_j+\bar{q}_j\right)\tilde{w}_j\,\mathrm{d}s+\int_\Gamma\left(-\frac{1}{2}[[\mu_{\underline{nn}}]]t_j+\bar{P}_j\right)u_j\,\mathrm{d}l=0 \tag{23}$$

The four integrands must vanish separately, so that the equilibrium equations are

$$\sigma_{ij,i}-\tfrac{1}{2}e_{ijk}\left(\mu_{pk,pi}+c_{k,i}\right)+f_j=0 \tag{24}$$

in Ω and the boundary conditions expressed in stresses are

$$\left(\sigma_{ij}-\tfrac{1}{2}e_{ijk}\left(\mu_{pk,p}+c_k-\mu_{\underline{nn},k}\right)\right)n_i=\bar{p}_j$$

$$\mu_{ij}n_i-\mu_{\underline{nn}}n_j=\bar{q}_j \tag{25}$$

and $\delta\Omega$ and

$$\tfrac{1}{2}[[\mu_{\underline{nn}}]]t_j=\bar{P}_j \tag{26}$$

along Γ. As anticipated, the boundary condition (25) contains only five equations.

From (25) and (26), it follows that the reduced contact loads on the surface, whether on the boundary or in the interior of Ω, are immediately retrievable from the stresses and couple-stresses.

2.5. Constitutive equations

For a linear isotropic elastic material, the potential energy-density is given by the quadratic form

$$W = W(\varepsilon_{ij}, \kappa_{ij}) = \frac{E}{2(1+v)}\left(\varepsilon_{ij}\varepsilon_{ij} + \frac{v}{1-2v}\varepsilon_{ii}\varepsilon_{jj} + 2l^2(\kappa_{ij}\kappa_{ij} + \eta\kappa_{ij}\kappa_{ji})\right) \tag{27}$$

where, along with the classical Young's modulus E and Poisson's ratio v, appear two additional material parameters: l homogeneous to a length and η dimensionless. From the condition of positive definiteness of W in the neighbourhood of the neutral state, it follows that

$$E > 0,\ l^2 > 0,\ -1 < v < 0.5 \ \text{and}\ -1 < \eta < 1 \tag{28}$$

so that l is indeed a length for being real and positive.

The stress–strain relations read

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \Longrightarrow \sigma_{ij} = \frac{E}{1+v}\left(\varepsilon_{ij} + \frac{v}{1-2v}\varepsilon_{kk}\delta_{ij}\right)$$

$$\mu_{ij} = \frac{\partial W}{\partial \kappa_{ij}} \Longrightarrow \mu_{ij} = \frac{2El^2}{1+v}(\kappa_{ij} + \eta\kappa_{ji}) \tag{29}$$

and their inversion is

$$\varepsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij}$$

$$\kappa_{ij} = \frac{1+v}{2El^2(1-\eta^2)}(\mu_{ij} - \eta\mu_{ji}) \tag{30}$$

3. Solution for pure bending

Since the couple-stress theory is a generalization of the classical continuum theory, the solution of the pure bending of a circular cross-section beam is sought as a generalization of the classical one. Specifically, to the classical solution will be superposed a corrective solution in plane strain within the cross-section. A cylindrical coordinate system will be used throughout, except for the calculation of the resultant and moment resultant on the section, which will be performed in a Cartesian coordinate system. Both coordinate systems are shown on Fig. 1 along with the bending axis.

3.1. Statement of the problem

The system of equations to be solved is composed of four groups of relations: the geometric relations (1), the stress–strain relations (29), the equilibrium eqns (24) and the boundary conditions (25) and (26). The last two groups of equations are now specified in the case of pure bending of a circular cylinder.

Since there are neither body forces nor body couples, the equilibrium equations reduce to

$$\sigma_{ij,i} - \tfrac{1}{2}e_{ijk}\mu_{pk,pi} = 0 \tag{31}$$

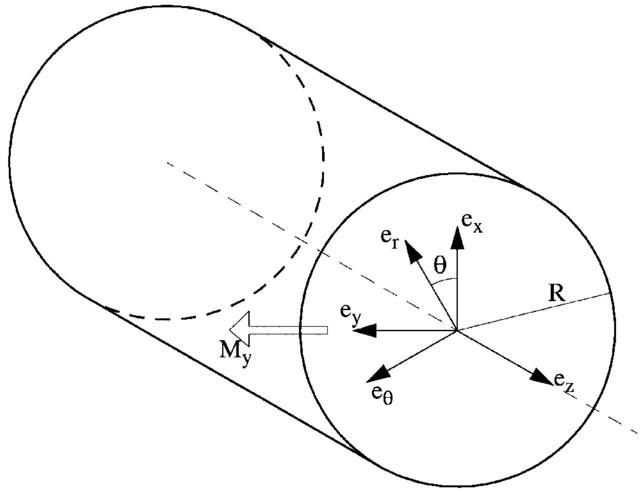

Fig. 1. Pure bending along e_y of a circular cross-section beam.

Since the lateral surface ($\boldsymbol{n} = \boldsymbol{e}_r$, and $r = R$) is free of forces and couples, the reduced surface loads are zero and the boundary conditions read

$$\sigma_{rj} - \tfrac{1}{2}e_{rjk}(\mu_{pk,p} - \mu_{rr,k}) = 0 \tag{32}$$

and

$$\mu_{rj} - \mu_{rr}\delta_{jr} = 0 \tag{33}$$

On the cross-section ($\boldsymbol{n} = \boldsymbol{e}_z$), the reduced surface forces and couples are given by

$$\bar{p}_j = \sigma_{zj} - \tfrac{1}{2}e_{zjk}(\mu_{pk,p} - \mu_{zz,k})$$

$$\bar{q}_j = \mu_{zj} - \mu_{zz}\delta_{jz} \tag{34}$$

On the edge of the cross-section ($\boldsymbol{n}^+ = \boldsymbol{e}_z$, $\boldsymbol{n}^- = \boldsymbol{e}_r$, $\boldsymbol{t} = \boldsymbol{e}_\theta$ and $r = R$), the reduced line forces are

$$\bar{P}_j = \tfrac{1}{2}(\mu_{zz} - \mu_{rr})\delta_{j\theta} \tag{35}$$

From (34) and (35), it follows that the resultant and moment resultant on the cross-section are

$$\mathscr{R}_x = \int_0^R \int_0^{2\pi} \big(\bar{p}_r \cos\theta - \bar{p}_\theta \sin\theta\big) r \, \mathrm{d}r \, \mathrm{d}\theta - \int_0^{2\pi} \bar{P}_\theta \sin\theta R \, \mathrm{d}\theta$$

$$\mathscr{R}_y = \int_0^R \int_0^{2\pi} \big(\bar{p}_r \sin\theta + \bar{p}_\theta \cos\theta\big) r \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \bar{P}_\theta \cos\theta R \, \mathrm{d}\theta$$

$$\mathscr{R}_z = \int_0^R \int_0^{2\pi} \bar{p}_z r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$\mathscr{M}_x = \int_0^R \int_0^{2\pi} \left(\bar{q}_r \cos\theta - \bar{q}_\theta \sin\theta\right) r \,\mathrm{d}r \,\mathrm{d}\theta + \int_0^R \int_0^{2\pi} \bar{p}_z \sin\theta r^2 \,\mathrm{d}r \,\mathrm{d}\theta$$

$$\mathscr{M}_y = \int_0^R \int_0^{2\pi} \left(\bar{q}_r \sin\theta + \bar{q}_\theta \cos\theta\right) r \,\mathrm{d}r \,\mathrm{d}\theta - \int_0^R \int_0^{2\pi} \bar{p}_z \cos\theta r^2 \,\mathrm{d}r \,\mathrm{d}\theta$$

$$\mathscr{M}_z = \int_0^R \int_0^{2\pi} \bar{p}_\theta r^2 \,\mathrm{d}r \,\mathrm{d}\theta + \int_0^{2\pi} \bar{P}_\theta R^2 \,\mathrm{d}\theta \tag{36}$$

For a state of pure bending around the y-axis, all the above components but $\mathscr{M}_y$ must vanish.

3.2. Classical solution

In cylindrical coordinates, the classical solution for pure bending is given by

$$u_r = \frac{\chi}{2}\left(vr^2 + z^2\right)\cos\theta$$

$$u_\theta = \frac{\chi}{2}\left(vr^2 - z^2\right)\sin\theta$$

$$u_z = -\chi rz \cos\theta \tag{37}$$

where χ is the curvature. According to (1), such a displacement field leads to the following strain and gradient of rotation:

$$\boldsymbol{\varepsilon} = \chi\begin{bmatrix} vr\cos\theta & 0 & 0 \\ 0 & vr\cos\theta & 0 \\ 0 & 0 & -r\cos\theta \end{bmatrix}, \quad \boldsymbol{\kappa} = \chi\begin{bmatrix} 0 & 0 & v\sin\theta \\ 0 & 0 & v\cos\theta \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \tag{38}$$

The associated stresses and couple-stresses are derived through the constitutive eqns (29):

$$\boldsymbol{\sigma} = E\chi\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r\cos\theta \end{bmatrix}, \quad \boldsymbol{\mu} = \chi\frac{2El^2}{1+v}\begin{bmatrix} 0 & 0 & (v+\eta)\sin\theta \\ 0 & 0 & (v+\eta)\cos\theta \\ (1+v\eta)\sin\theta & (1+v\eta)\cos\theta & 0 \end{bmatrix} \tag{39}$$

The divergence of the (non symmetric) couple-stress tensor, i.e.

$$\mu_{pk,p} = \begin{bmatrix} \dfrac{\partial\mu_{rr}}{\partial r} + \dfrac{1}{r}\left(\mu_{rr} - \mu_{\theta\theta} + \dfrac{\partial\mu_{\theta r}}{\partial\theta}\right) + \dfrac{\partial\mu_{zr}}{\partial z} \\ \dfrac{\partial\mu_{r\theta}}{\partial r} + \dfrac{1}{r}\left(\mu_{r\theta} + \mu_{\theta r} + \dfrac{\partial\mu_{\theta\theta}}{\partial\theta}\right) + \dfrac{\partial\mu_{z\theta}}{\partial z} \\ \dfrac{\partial\mu_{rz}}{\partial r} + \dfrac{1}{r}\left(\mu_{rz} + \dfrac{\partial\mu_{\theta z}}{\partial\theta}\right) + \dfrac{\partial\mu_{zz}}{\partial z} \end{bmatrix} \tag{40}$$

and the component μ_{rr} turn out to vanish. Thus the equilibrium eqns (31) and the first boundary conditions on the lateral surface (32) reduce to their classical expressions and are therefore satisfied by the classical stresses $\boldsymbol{\sigma}$. Unfortunately, the second boundary conditions on the lateral surface (33) are violated unless $\eta = -\nu$ since

$$\mu_{rj} - \mu_{rr}\delta_{jr} = \mu_{rj} = 2\chi E l^2 \frac{\nu + \eta}{1 + \nu} \sin\theta\delta_{zj} \tag{41}$$

Contrarily to the case of torsion (Koiter, 1964; Sokolowski, 1970), the classical solution needs here a modification in order to comply with the couple-stress theory. The same situation is encountered for the pure bending of a rectangular cross-section as quoted by Koiter (1964). As a matter of fact, this author did not attempt to obtain the exact solution but derived upper and lower bounds of the flexural rigidity.

3.3. Additional solution in plane strain

The additional solution is sought in plane strain in the section perpendicular to the z-axis, with the aid of the generalized Airy stress functions introduced by Mindlin (1963). This method is most effective here because the problem to be solved is formulated in stresses and the determination of displacements could be avoided. However, for the sake of completeness, the displacements will also be derived.

In a state of plane strain in the plane perpendicular to the z-axis, the stresses and couple-stresses solution of eqns (1), (29) and (31) are given by

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}$$

$$\sigma_{zz} = \nu\left(\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial r^2}\right)$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{2}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)$$

$$\sigma_{rz} = \sigma_{zr} = \sigma_{\theta z} = \sigma_{z\theta} = 0$$

$$\mu_{rz} = \frac{\partial\psi}{\partial r}$$

$$\mu_{\theta z} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\mu_{zr} = \eta\frac{\partial\psi}{\partial r}$$

$$\mu_{z\theta} = \eta\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\mu_{rr} = \mu_{\theta\theta} = \mu_{r\theta} = \mu_{\theta r} = \mu_{zz} = 0 \tag{42}$$

where ϕ and ψ are generalized Airy stress functions related to each other by

$$\frac{\partial}{\partial r}\left(\psi - l^2\nabla^2\psi\right) = -2(1-v)l^2\frac{1}{r}\frac{\partial}{\partial\theta}\nabla^2\phi$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(\psi - l^2\nabla^2\psi\right) = 2(1-v)l^2\frac{\partial}{\partial r}\nabla^2\phi \tag{43}$$

The derivation of the stress functions satisfying (43) is facilitated by noting that they necessarily satisfy

$$\nabla^2\nabla^2\phi = 0$$

$$\nabla^2\psi - l^2\nabla^2\nabla^2\psi = 0 \tag{44}$$

Thus, for the sought additional solution, we take the following stress functions

$$\phi = Ar^3\cos\theta$$

$$\psi = \left(BI_1\left(\frac{r}{l}\right) + Cr\right)\sin\theta \tag{45}$$

where A, B and C are three constants and I_n is the modified Bessel function of the first kind and order n. It is reminded that:

$$\frac{d}{dr}I_1\left(\frac{r}{l}\right) = \frac{1}{r}I_1\left(\frac{r}{l}\right) + \frac{1}{l}I_2\left(\frac{r}{l}\right)$$

$$\frac{d^2}{dr^2}I_1\left(\frac{r}{l}\right) = \frac{1}{l^2}I_1\left(\frac{r}{l}\right) - \frac{1}{rl}I_2\left(\frac{r}{l}\right) \tag{46}$$

Relations (43) are satisfied if

$$C = 16(1-v)l^2A \tag{47}$$

From expression (42), the non-vanishing components of the stresses and couple-stresses are

$$\sigma_{rr} = \left(2Ar - \frac{B}{rl}I_2\left(\frac{r}{l}\right)\right)\cos\theta$$

$$\sigma_{\theta\theta} = \left(6Ar + \frac{B}{rl}I_2\left(\frac{r}{l}\right)\right)\cos\theta$$

$$\sigma_{zz} = 8Avr\ \cos\theta$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \left[2Ar + \frac{B}{2rl}\left(\frac{r}{l}I_1\left(\frac{r}{l}\right) - 2I_2\left(\frac{r}{l}\right)\right)\right]\sin\theta$$

$$\mu_{rz} = \left[\frac{B}{r}\left(I_1\left(\frac{r}{l}\right) + \frac{r}{l}I_2\left(\frac{r}{l}\right)\right) + C\right]\sin\theta$$

$$\mu_{\theta z} = \left(\frac{B}{r}I_1\left(\frac{r}{l}\right) + C\right)\cos\theta$$

$$\mu_{zr} = \eta\left[\frac{B}{r}\left(I_1\left(\frac{r}{l}\right) + \frac{r}{l}I_2\left(\frac{r}{l}\right)\right) + C\right]\sin\theta$$

$$\mu_{z\theta} = \eta\left(\frac{B}{r}I_1\left(\frac{r}{l}\right) + C\right)\cos\theta \tag{48}$$

It remains to check the boundary conditions. The divergence of the couple-stress tensor being

$$\mu_{pk,p} = \frac{B}{l^2}I_1\left(\frac{r}{l}\right)\sin\theta\delta_{zk} \tag{49}$$

and the component μ_{rr} being zero, the first boundary condition on the lateral surface (32) reduces to

$$\sigma_{r\theta} - \frac{B}{2l^2}I_1\left(\frac{R}{l}\right)\sin\theta = 0 \tag{50}$$

and, owing to (48), is satisfied if

$$A = \frac{B}{2R^2 l}I_2\left(\frac{R}{l}\right) \tag{51}$$

In the second boundary condition on the lateral surface (33), the error (41) introduced by the classical solution should be removed, so that we obtain

$$\mu_{rj} = -2\chi El^2\frac{v+\eta}{1+v}\sin\theta\delta_{zj} \tag{52}$$

which, owing to (48), is satisfied if

$$\frac{B}{R}\left(I_1\left(\frac{R}{l}\right) + \frac{R}{l}I_2\left(\frac{R}{l}\right)\right) + C = -2\chi El^2\frac{v+\eta}{1+v} \tag{53}$$

Relations (47), (51) and (53) allow us to determine the constants A, B and C. Putting $\alpha = (R/l)$, one gets:

$$B = \frac{-2\chi ER^3(v+\eta)}{(1+v)\alpha\left[\alpha I_1(\alpha) + (8(1-v)+\alpha^2)I_2(\alpha)\right]}$$

$$A = \frac{\alpha I_2(\alpha)}{2R^3}B$$

$$C = \frac{8(1-v)I_2(\alpha)}{\alpha R}B \tag{54}$$

The additional solution is thus fully determined in terms of stresses. The corresponding displacements may be easily derived by integrating the geometric relations (1) with the strain and gradient of rotation computed from (48) through relations (30). The result is:

$$u_r = \frac{1+v}{E}\left((1-4v)Ar^2 - \frac{B}{r}I_1\left(\frac{r}{l}\right)\right)\cos\theta$$

$$u_\theta = \frac{1+v}{E}\left((5-4v)Ar^2 + \frac{B}{r}\left(I_1\left(\frac{r}{l}\right) + \frac{r}{l}I_2\left(\frac{r}{l}\right)\right)\right)\sin\theta$$

$$u_z = 0 \tag{55}$$

3.4. Complete solution and bending inertia

The complete solution is the sum of the classical solution (37) and the additional solution in plane strain (55), i.e. after substitution of A, B and C according to (54),

$$u_r = \chi\left\{\left[\frac{v}{2} - \frac{(v+\eta)(1-4v)I_2(\alpha)}{\alpha I_1(\alpha) + (8(1-v)+\alpha^2)I_2(\alpha)}\right]r^2 + \frac{2(v+\eta)R^2\frac{l}{r}I_1\left(\frac{r}{l}\right)}{\alpha I_1(\alpha) + (8(1-v)+\alpha^2)I_2(\alpha)} + \frac{z^2}{2}\right\}\cos\theta$$

$$u_\theta = \chi\left\{\left[\frac{v}{2} - \frac{(v+\eta)(5-4v)I_2(\alpha)}{\alpha I_1(\alpha) + (8(1-v)+\alpha^2)I_2(\alpha)}\right]r^2 - \frac{2(v+\eta)R^2\left[\frac{l}{r}I_1\left(\frac{r}{l}\right) + I_2\left(\frac{r}{l}\right)\right]}{\alpha I_1(\alpha) + (8(1-v)+\alpha^2)I_2(\alpha)} - \frac{z^2}{2}\right\}\sin\theta$$

$$u_z = -\chi rz\ \cos\theta \tag{56}$$

The stresses and couple-stresses being obtained by the summation of (39) and (48), the reduced loads $\bar{\boldsymbol{p}}$, $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{P}}$ on the cross-section may be now computed according to formulae (34) and (35). Taking into account that the components μ_{zz} and μ_{rr} are zero and that the divergence of the couple stress tensor is still given by eqn (49), the non-vanishing components of the reduced loads are:

$$\bar{p}_z = (8Av - E\chi)r\ \cos\theta$$

$$\bar{q}_r = \left[2\chi El^2\frac{1+v\eta}{1+v} + \eta\frac{B}{r}\left(I_1\left(\frac{r}{l}\right) + \frac{r}{l}I_2\left(\frac{r}{l}\right)\right) + \eta C\right]\sin\theta$$

$$\bar{q}_\theta = \left(2\chi El^2\frac{1+v\eta}{1+v} + \eta\frac{B}{r}I_1\left(\frac{r}{l}\right) + \eta C\right)\cos\theta \tag{57}$$

Substituting (57) into (36) and performing the required integrations, it turns out that the only non-vanishing component is $\mathscr{M}_y$:

$$\mathscr{M}_y = \left(2\chi El^2\frac{1+v\eta}{1+v} + \eta C\right)\pi R^2 + \eta BI_1(\alpha)\pi R + (E\chi - 8Av)\frac{\pi R^4}{4} \tag{58}$$

Substituting the values of the constants A, B and C from (54), one finally gets

$$\mathscr{M}_y = \frac{\chi E\pi R^4}{4}\left\{1 + \frac{8(1-\eta^2)}{\alpha^2(1+v)} + \frac{8(v+\eta)^2 I_2(\alpha)}{(1+v)\left[\alpha I_1(\alpha) + (8(1-v)+\alpha^2)I_2(\alpha)\right]}\right\} \tag{59}$$

It is recalled that the classical bending inertia is $I_c = (\pi R^4/4)$. Then the new one is written as

$$I = I_c\left\{1 + \frac{8(1-\eta^2)}{\alpha^2(1+v)} + \frac{8(v+\eta)^2 I_2(\alpha)}{(1+v)\left[\alpha I_1(\alpha) + (8(1-v)+\alpha^2)I_2(\alpha)\right\}}\right\} \tag{60}$$

This formula is a generalization of the classical one: when $l \to 0$, that is when $\alpha \to +\infty$, both functions $I_1(\alpha)$ and $I_2(\alpha)$ are equivalent to $(e^\alpha/\sqrt{e\pi\alpha})$ so that

$$I \sim I_c\left[1 + \frac{8(1-\eta^2)}{\alpha^2(1+v)} + \frac{8(v+\eta)^2}{\alpha^2(1+v)}\right] = I_c\left[1 + 8\frac{1-\eta^2+(v+\eta)^2}{1+v}\left(\frac{l}{R}\right)^2\right] \tag{61}$$

The asymptotic expression (61) proves that the classical bending inertia is retrieved when $l \to 0$.

The new bending inertia is always larger than the classical one, provided that conditions (28) are satisfied. In particular, when $\eta = -v$, the complete solution reduces to the classical one and the formula (60) takes a particularly simple expression

$$I = I_c\left\{1 + \frac{8(1-v)}{\alpha^2}\right\} \tag{62}$$

Considering that, in polycrystalline metal or granular material, l is probably of the order of the dimension of the crystals or grains, the ratio $\alpha = R/l$ cannot take very small values. In Fig. 2, the variations of I/I_c are plotted against η and α, for $v = 0.3$. In Fig. 3, they are plotted against η and v, for $\alpha = 4$. In Fig. 4, the variations of I/I_c are plotted against η, for $v = 0.3$ and $\alpha = 4$, 5, 10, 20, 50. In practice, the influence of the couple-stresses diminished rapidly as α increases: independently of $v \geq 0$ and η, the relative difference between I and I_c is less than 3% as soon as α is greater than 20.

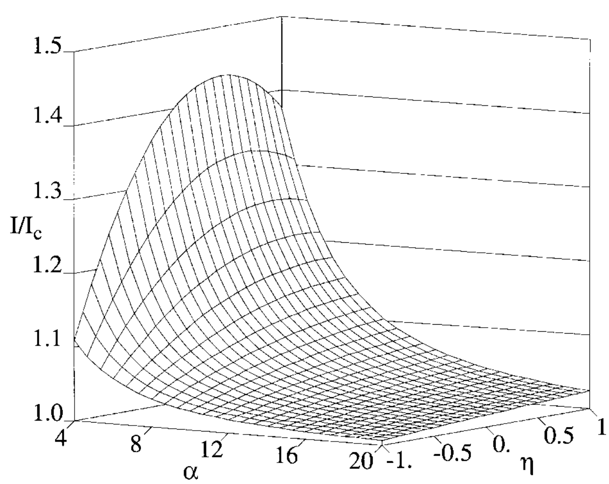

Fig. 2. Bending inertia of a circular cross-section for $v = 0.3$: variation with η and α.

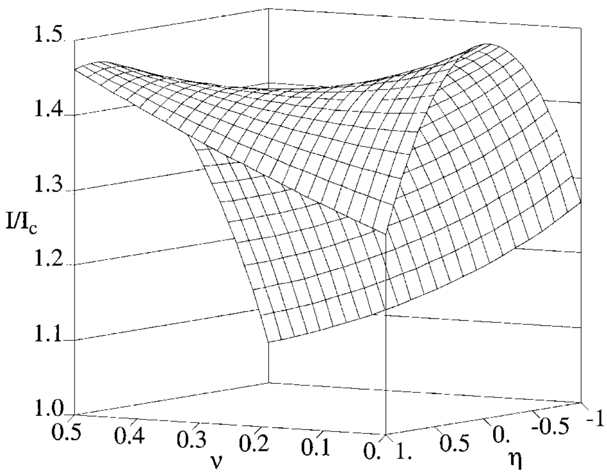

Fig. 3. Bending inertia of a circular cross-section for $\alpha = 4$: variation with ν and η.

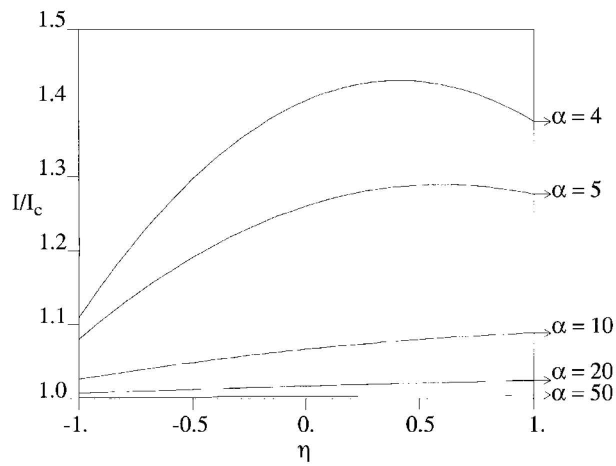

Fig. 4. Bending inertial of a circular cross-section for $\nu = 0.3$ and $\alpha = 4, 5, 10, 20, 50$: variation with η.

The linear couple-stress theory might therefore be checked by performing four point bending tests on circular bars of various (possible small) radii: the bending rigidity EI is then the ratio of the moment over the curvature between the load points, i.e. $EI = PLd^2/2\Delta f$ is the difference between the deflections of the mid-point and the point under the force P (Fig. 5). If formula (60) holds true, the ratio EI/R^4 should not be constant but should increase for decreasing values of the radius.

4. Conclusion

The complete solution for the pure bending of a circular cylinder has been derived within the couple-stress theory where only the gradient of rotation is taken into account. The recalculation of the bending

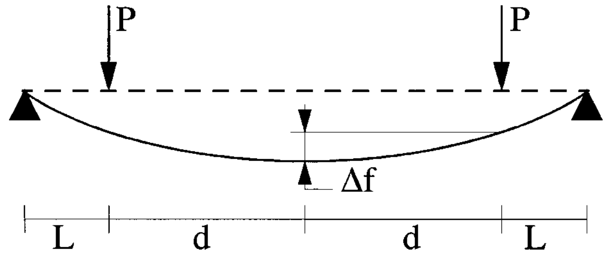

Fig. 5. Four point bending test.

inertia of a circular cross-section results in higher values than those accepted before, especially when the ratio of the radius of the beam to the characteristic material length l is lower than 20.

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